

# Zeta function regularization, anomaly and complex mass term

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## Abstract

If the zeta function regularization is used and a complex mass term considered for fermions, the phase does not appear in the fermion determinant. This is not a drawback of the regularization, which can recognize the phase through source terms, as demonstrated by the anomaly equation which is explicitly derived here for a complex mass term.

## 1 Introduction

The predictions of a field theory can be formally expressed in terms of functional integrals. For the fermionic sector, one writes

$$\int d\mu \exp\left[\int \bar{\psi} \mathcal{D} \psi\right] = \det \mathcal{D}, \quad (1)$$

where  $\mathcal{D}$  stands for the appropriate Dirac operator for the case. Apart from a kinetic part, the interaction with gauge fields is also included. The mass term is often taken to be the simple  $-\int \bar{\psi} m \psi$ , but the possibility of CP violation requires the consideration of what is sometimes called a twisted mass or simply a complex mass [1], namely  $-\int \bar{\psi} m \exp(i\theta \gamma^5) \psi$ . It is this  $\theta$ -term which will be of central interest in the following.

At the classical level, the phase  $\theta$  may be removed by a chiral transformation, but in the full quantum field theory, the situation is more complicated because of the chiral anomaly. It is well known now that the fermion measure is not invariant under a chiral transformation, so that an attempt to remove  $\theta$  by such a transformation may produce a non-trivial Jacobian dependent on  $\theta$ , thus causing the reappearance of this parameter. However, the anomaly is a result of short distance singularities, and needs to be studied with a proper regularization. We shall discuss the consequences of the  $\theta$ -term in the context of a specific regularization, the zeta function regularization. This approach has been shown to yield a functional integral independent of  $\theta$ , which may suggest a limitation of the approach. However, the phase has to appear in Green functions and it duly does, when these are calculated by using sources. The anomaly equation

too has to contain  $\theta$ , not in the anomaly but in the classical mass term. It is not obvious that this will occur in the zeta function framework which makes use of the  $\theta$ -independent Laplacian operator. Hence we study the anomaly equation in the case of a complex mass term and verify that the regularization is capable of accommodating the phase as well as the anomaly. This demonstrates the reliability of the zeta function regularization and makes the use of this approach to the regularization of the determinant acceptable even in this case.

## 2 Review of fermion determinant in zeta function regularization

The determinant of a matrix can be thought of as the product of its eigenvalues. For an operator, the product of the eigenvalues has to be regularized.

The zeta function regularization is widely used in mathematical discussions in quantum field theory [2, 3, 4]. It was shown quite a while back that the chiral anomaly in vector gauge theories can be evaluated by using this regularization without recourse to Feynman diagrams [5].

The zeta function of an operator  $X$  involves a parameter  $s$ ,

$$\zeta(s, X) \equiv \text{Tr}(X^{-s}). \quad (2)$$

In terms of eigenvalues  $\lambda$ , this becomes

$$\zeta(s, X) = \sum (\lambda^{-s}), \quad (3)$$

so that

$$\zeta'(s, X) = - \sum (\ln \lambda \lambda^{-s}), \quad (4)$$

and

$$\zeta'(0, X) = - \sum (\ln \lambda) = - \ln \prod \lambda = - \ln \det X. \quad (5)$$

This provides a definition of the determinant. The eigenvalues are assumed to be positive in this definition.

Note that the Dirac operator

$$\mathcal{D} = i \not{D} - m \exp(i\theta\gamma^5) \quad (6)$$

is neither hermitian nor antihermitian even for a real mass term. A *positive* operator is constructed for the zeta function by going over to the Laplacian from the Dirac operator as in [5]:

$$\Delta = [i \not{D} - m \exp(i\theta\gamma^5)]^\dagger [i \not{D} - m \exp(i\theta\gamma^5)]. \quad (7)$$

Formally, the two factors are related by chiral transformations and formally the determinants of all these transformations can be taken to be unity, so the

determinant may be said to have been squared in the process, and a square root has to be included in the definition of the determinant.

For *antihermitian*  $\gamma$ -matrices, as appropriate for euclidean spacetime,  $\Delta$  is independent of the phase  $\theta$ :

$$\Delta = (\not{D})^2 + m^2. \quad (8)$$

The zeta function of this operator is

$$\zeta(s, \Delta) \equiv \text{Tr}(\Delta^{-s}), \quad (9)$$

and the regularized logarithm of the functional integral is defined in the limit of  $s \rightarrow 0$  as

$$\ln Z \equiv -\frac{1}{2}\zeta'(0, \Delta) - \frac{1}{2} \ln \mu^2 \zeta(0, \Delta). \quad (10)$$

The square root is introduced because of the squaring in the construction of  $\Delta$  mentioned above. It is to be noted that the determinant is defined only for the product  $\Delta$  and not for the Dirac operators.

The regularized determinant is independent of  $\theta$ , depending on the gauge fields only through the operator  $\Delta$  and is therefore invariant under symmetry transformations of the gauge field  $A$  [6].

This may be compared with the formal determinant of a Dirac operator when  $\not{D}$  has only a finite number of zero modes and no other eigenvalue. Because of the anticommutation of  $\not{D}$  and  $\gamma^5$ , the zero modes can be chosen to be of definite chirality. The mass term produces a factor of  $\exp(i\theta\gamma^5)$  for each zero mode, leading to a product  $\exp(i\theta\nu)$  where  $\nu$  stands for the number of positive chirality zero modes *reduced* by the number of negative chirality zero modes. This number depends on the gauge field involved in  $D_\mu$ . Such a factor will continue to appear if a finite number of nonzero modes occur. However, when the number of modes becomes infinite, the reordering of the eigenvalues involved in identifying such a factor is not admissible and regularization is crucial. A regularization may even remove the  $\theta$  dependence. But it has to be checked whether the regularization vitiates the anomaly equation.

### 3 Inclusion of fermion sources

If one wants to calculate fermion Green functions, one has to introduce fermion source terms in the standard way. This means the consideration of

$$\int d\mu \exp\left[\int (\bar{\psi}\not{D}\psi + \bar{\psi}\eta + \bar{\eta}\psi)\right] = \det\not{D} \exp\left[-\int \bar{\eta}\not{D}^{-1}\eta\right]. \quad (11)$$

The determinant is defined by the zeta function method indicated above, while the source dependent factor is separate. This factor explicitly involves  $\theta$  through  $\not{D}$ . Thus fermionic Green functions continue to depend on  $\theta$  in the zeta function approach. This is because the fermion field is not chiral invariant. There is no

contribution to the  $\theta$  dependence from fermion loops in the determinant. As an example, the propagator is given by

$$\langle \psi(x) \bar{\psi}(y) \rangle = \langle \mathcal{D}^{-1}(x, y) \rangle, \quad (12)$$

where the averaging is over gauge fields with the effective gauge field action arising from the original gauge field action and including the effect of the fermion determinant. This depends on  $\theta$  through  $\mathcal{D}^{-1}$ . The only  $\theta$  dependence however goes away when the external legs in a Feynman diagram are amputated, *i.e.*, the  $\mathcal{D}^{-1}$ -s are removed by  $\mathcal{D}$ .

## 4 Anomaly equation and $\theta$

The  $\theta$ -independence of the determinant may suggest that the zeta function approach, relying as it does on the product of the Dirac operator with its conjugate, is somewhat handicapped and not sensitive to the presence of  $\theta$ . If this were the case, it would be a serious problem for the zeta function approach. It has already been pointed out that Green functions do contain  $\theta$ . The crucial anomaly equation also has to involve the parameter  $\theta$ :

$$\partial^\mu \langle \bar{\psi} \gamma_\mu \gamma_5 \psi \rangle = 2im \langle \bar{\psi} \gamma_5 \exp(i\theta \gamma^5) \psi \rangle + \text{anomaly}.$$

This is particularly significant because the possibility of obstructions to the removal of  $\theta$  from the fermion action arises entirely from the anomaly. Although the anomaly has been demonstrated in this approach, it was for a real mass term, where  $\theta = 0$ . The anomaly is supposed to be independent of the mass, so one can expect it to arise also for a complex mass term, but does  $\theta$  appear as indicated in the zeta function approach?

To derive the anomaly equation in this framework, one has to add source terms for the composite fermion operators in that equation to the Dirac operator [5]. The phase  $\theta$  requires us to consider the modified operator

$$[i \not{D} - i \not{Q} \gamma^5 - m \exp(i\theta \gamma^5) - K \gamma^5 \exp(i\theta \gamma^5)], \quad (13)$$

with  $Q^\mu(x)$  coupling to the axial current and  $K$  to the pseudoscalar density including the phase. Note that in the presence of the phase in the mass term, the parity symmetry transformation of the action is chirally rotated, so that  $\bar{\psi} \exp(i\theta \gamma^5) \psi$  is a scalar and  $\bar{\psi} \gamma^5 \exp(i\theta \gamma^5) \psi$  is a pseudoscalar. That is why the pseudoscalar source has to have the chiral phase factor. This leads to the modified Laplacian

$$\begin{aligned} \Delta' &= [i \not{D} - i \not{Q} \gamma^5 - m \exp(i\theta \gamma^5) - K \gamma^5 \exp(i\theta \gamma^5)]^\dagger \\ &\quad [i \not{D} - i \not{Q} \gamma^5 - m \exp(i\theta \gamma^5) - K \gamma^5 \exp(i\theta \gamma^5)] \\ &= (\not{D})^2 + m^2 + K^2 - (\not{Q})^2 - \not{Q} \gamma^5 \not{D} - \not{D} \not{Q} \gamma^5 + 2mK \gamma^5 \\ &\quad + i(\not{D} - \not{Q} \gamma^5)K \gamma^5 \exp(i\theta \gamma^5) - iK \gamma^5 \exp(-i\theta \gamma^5)(\not{D} - \not{Q} \gamma^5), \end{aligned} \quad (14)$$

which does depend on  $\theta$ . This  $\Delta'$  operator is used to define a modified  $Z'$  and thence the expectation value of the axial current operator

$$\langle \bar{\psi} \gamma_\mu \gamma_5 \psi \rangle = i \frac{\delta \ln Z'}{\delta Q^\mu(x)}|_{Q=K=0} = -\frac{i}{2} \frac{\delta \zeta'(0, \Delta')}{\delta Q^\mu(x)}|_{Q=K=0}. \quad (15)$$

Now let  $\phi_n$  be eigenfunctions and  $\lambda_n$  the corresponding eigenvalues for  $\Delta$ , primed as required, but see below:

$$\Delta \phi_n = \lambda_n \phi_n, \quad \Delta' \phi'_n = \lambda'_n \phi'_n. \quad (16)$$

Then by the Hellman - Feynman theorem,

$$\begin{aligned} \frac{\delta \zeta(s, \Delta')}{\delta Q^\mu(x)}|_{Q=K=0} &= \sum_n \frac{\delta \lambda_n'^{-s}}{\delta Q^\mu(x)}|_{Q=K=0} \\ &= -s \sum_n \lambda_n'^{-s-1} \frac{\delta \lambda_n'}{\delta Q^\mu(x)}|_{Q=K=0} \\ &= -s \sum_n \lambda_n'^{-s-1} \int d^4 w \phi_n^\dagger(w) \frac{\delta \Delta'}{\delta Q^\mu(x)}|_{Q=K=0} \phi_n(w) \\ &= s \sum_n \lambda_n'^{-s-1} \phi_n^\dagger(x) [\gamma_\mu \gamma^5 \not{D} - \overleftarrow{\not{D}} \gamma_\mu \gamma^5] \phi_n(x). \end{aligned} \quad (17)$$

On taking the divergence one can simplify the expression if one takes the  $\phi_n$  to be eigenfunctions of  $\exp(-i\theta\gamma^5) \not{D}$  in addition to  $\Delta = (\not{D})^2 + m^2$ .

$$\exp(-i\theta\gamma^5) \not{D} \phi_n = \alpha_n \phi_n. \quad (18)$$

This is possible with

$$\alpha_n^2 + m^2 = \lambda_n \quad (19)$$

because  $\exp(-i\theta\gamma^5) \not{D}$  is hermitian and its square is the same as  $(\not{D})^2$ , which differs only by  $m^2$  from  $\Delta$ . The exponential factor is not relevant here, it just cancels out on squaring the operator and does not cause any problem; however, it is needed at a later stage. One finds

$$\begin{aligned} \partial^\mu \langle \bar{\psi} \gamma_\mu \gamma_5 \psi \rangle &= 2i \sum_n [s \lambda_n'^{-s} \phi_n^\dagger(x) \gamma^5 \phi_n(x)]'|_{s \rightarrow 0} \\ &\quad - 2im^2 \sum_n [s \lambda_n'^{-s-1} \phi_n^\dagger(x) \gamma^5 \phi_n(x)]'|_{s \rightarrow 0}. \end{aligned} \quad (20)$$

The first term of (20) is reminiscent of  $2i \sum_n \exp(-\lambda_n/M^2) \phi_n^\dagger(x) \gamma^5 \phi_n(x)$  in the measure approach and similar calculations [5] show it to be the anomaly term

$$\frac{i}{16\pi^2} \text{tr} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (21)$$

The second term of (20) does not depend explicitly on the phase  $\theta$  but the eigenfunctions  $\phi_n$  implicitly involve it. To rewrite the term in a more familiar form, one has to recognize

$$\langle \bar{\psi} \gamma_5 \exp(i\theta \gamma^5) \psi \rangle = -\frac{\delta \ln Z'}{\delta K(x)}|_{Q=K=0} = \frac{1}{2} \frac{\delta \zeta'(0, \Delta')}{\delta K(x)}|_{Q=K=0}. \quad (22)$$

Now as in the case of the axial vector above, one has

$$\begin{aligned} \frac{\delta \zeta(s, \Delta')}{\delta K(x)}|_{Q=K=0} &= \sum_n \frac{\delta \lambda_n'^{-s}}{\delta K(x)}|_{Q=K=0} \\ &= -s \sum_n \lambda_n^{-s-1} \frac{\delta \lambda_n'}{\delta K(x)}|_{Q=K=0} \\ &= -s \sum_n \lambda_n^{-s-1} \int d^4 w \phi_n^\dagger(w) \frac{\delta \Delta'}{\delta K(x)}|_{Q=K=0} \phi_n(w) \\ &= -s \sum_n \lambda_n^{-s-1} \phi_n^\dagger(x) [-i\gamma^5 \exp(-i\theta \gamma^5) \not{D} \\ &\quad -i \overleftarrow{\not{D}} \gamma^5 \exp(i\theta \gamma^5) + 2m\gamma^5] \phi_n(x). \end{aligned} \quad (23)$$

This may be simplified by using the fact that  $\phi_n(x)$  is an eigenfunction of the operator  $\exp(-i\theta \gamma^5) \not{D}$  in addition to  $\Delta = (\not{D})^2 + m^2$ , as mentioned earlier. One finds

$$\langle \bar{\psi} \gamma_5 \exp(i\theta \gamma^5) \psi \rangle = -m \sum_n [s \lambda_n^{-s-1} \phi_n^\dagger(x) \gamma^5 \phi_n(x)]'|_{s \rightarrow 0}, \quad (24)$$

so that

$$\partial^\mu \langle \bar{\psi} \gamma_\mu \gamma_5 \psi \rangle = 2im \langle \bar{\psi} \gamma_5 \exp(i\theta \gamma^5) \psi \rangle + \frac{i}{16\pi^2} \text{tr} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (25)$$

This is the anomaly equation in euclidean space and it contains  $\theta$  as expected in the mass term. This confirms that the zeta function approach is not blind to  $\theta$  even though the determinant is.

## 5 Conclusion

To conclude, we have reproduced the anomaly equation for a complex mass term using the zeta function approach. The anomaly assures us that the  $\theta$  independence of the determinant in the zeta function approach is not due to any inability to perceive the anomaly. Apart from the anomaly, there is the non-anomalous mass-dependent piece in the equation which has to contain the phase when the mass term in the action contains it. This is what has been checked by generalizing the derivation for real mass terms. In doing so, the pseudoscalar source and the eigenfunctions have had to be reorganized because of the phase. The importance of the calculation is that it shows that the use of

the Laplacian operator, which is *independent of the phase*, does not compromise the power of the zeta function approach and thus makes this approach acceptable even in the context of complex mass terms.

## References

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